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Stationary isothermic surfaces of the heat flow

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS 3]. Let $u=u(x,t)$ be the unique solution of the following problem for the heat equation:

$$\partial_t u=\Delta u \quad \text{in} \quad \Omega\times(0,+\infty), \tag{1.1}$$

$$u=1 \quad \text{on} \quad \partial\Omega\times(0,+\infty), \tag{1.2}$$

$$u=0 \quad \text{on} \quad \Omega\times\{0\}, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 2$.

A conjecture, posed in [Kl] by M.S. Klamkin and referred to by L. Zalcman in [Z] as the *Matzoh Ball Soup*, was settled affirmatively by G. Alessandrini in [A 1]-[A 2]. In [A 2], under the assumption that every point of $\partial\Omega$ is regular with respect to the Laplacian, it was proved that if all the spatial isothermic surfaces of u are *invariant with time* then Ω must be a *ball*. (Of course, the values of u vary with time on its spatial isothermic surfaces.)

The case where the homogeneous initial data in (1.3) is replaced by a function in the space $L^2(\Omega)$ was also considered in [A 1]-[A 2] and, with the help of J. Serrin's celebrated symmetry theorem for elliptic equations [Ser], was settled in the following terms: if all the spatial isothermic surfaces of the solution u of the heat equation with homogeneous Dirichlet boundary condition and initial data $\varphi\in L^2(\Omega)$ are invariant with time, then either φ is an eigenfunction of the Laplacian or Ω is a ball. The analogous question where condition (1.2) is replaced by the homogeneous Neumann boundary condition was examined and answered positively (see [Sak], Theorem 1) with the aid of the classification theorem for *isoparametric hypersurfaces in Euclidean* *space* due to T. Levi-Civita and B. Segre (see [LC], [Seg]). The method used in [Sak] can be applied to give an alternative proof of Alessandrini's results.

An important observation is that, in order to prove Klamkin's conjecture [Kl], both methods employed in [A 1]-[A 2] and [Sak] need to assume that *infinitely many* isothermic surfaces of u are invariant with time. As a natural consequence of this remark, one may wonder if the requirement that a finite number (possibly only one) of level surfaces of u are invariant with time implies that Ω is a ball.

Our main result in this direction is the following.

Theorem 1.1 *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is a domain, with boundary ∂D, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$. Assume that the solution u of problem (1.1)-(1.3) satisfies the following condition:*

$$u(x,t) = a(t), \quad (x,t) \in \partial D \times (0, +\infty), \tag{1.4}$$

for some function $a : (0, +\infty) \to (0, +\infty)$. Then Ω must be a ball.

We recall that Ω satisfies the *exterior sphere condition* if for every $y \in \partial\Omega$ there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, where $B_r(z)$ denotes an open ball centered at $z \in \mathbb{R}^N$ and with radius $r > 0$. Also, D satisfies the *interior cone condition* if for every $x \in \partial D$ there exists a finite right spherical cone K_x with vertex x such that $K_x \subset \overline{D}$ and $\overline{K_x} \cap \partial D = \{x\}$.

The proof of Theorem 1.1 exploits arguments different from the ones used in [A 1]-[A 2] and [Sak]. Our technique is essentially based on the following three ingredients. One ingredient is a careful study of the asymptotic behavior of $u(x,t)$ as $t \to 0$ which is based on the results of S. R. S. Varadhan [V] (see also [EI]). The second one is A. D. Aleksandrov's uniqueness theorem [Alek]. A special case of this theorem is the well-known *Soap-Bubble Theorem.* The third one is the following balance law proved in [MS 1]-[MS 2] (see [MS 3] for a shorter proof):

Theorem 1.2 (balance law) *Let G be a domain in $\mathbb{R}^N$, $N \geq 2$, let x_0 be a point in G and set $d_* = \operatorname{dist}(x_0, \partial G)$. Suppose that $v = v(x,t)$ is a solution of the heat equation in $G \times (0, +\infty)$. Then the following hold:*

(i) $v(x_0, t) = 0$ *for every* $t \in (0, +\infty)$ *if and only if*

$$\int_{\partial B_r(x_0)} v(x,t)\ dS_x = 0 \text{ for every } (r,t) \in (0, d_*) \times (0, +\infty);$$

(ii) $\nabla v(x_0, t) = 0$ *for every* $t \in (0, +\infty)$ *if and only if*

$$\int_{\partial B_r(x_0)} (x - x_0) v(x,t)\ dS_x = 0 \text{ for every } (r,t) \in (0, d_*) \times (0, +\infty).$$

Section 2 is devoted to an outline of the proof of Theorem 1.1. In Section 3, we consider the case where the domain Ω is unbounded.

2 Outline of the proof of Theorem 1.1

Define the function $W = W(x, s)$ by

$$W(x,s) = s \int_0^{+\infty} u(x,t)\ e^{-s\ t} dt, \ \ s > 0. \tag{2.1}$$

Notice that W is the solution of the following elliptic boundary value problem:

$$\Delta W - s\, W = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$W = 1 \quad \text{on } \partial\Omega. \tag{2.3}$$

A result in [V] (see also [EI]) shows that, as $s \to +\infty$, the function $-\frac{1}{\sqrt{s}} \log W(x,s)$ converges uniformly on $\overline{\Omega}$ to the function $d = d(x)$ defined by

$$d(x) = \text{dist}\ (x, \partial\Omega), \ \ x \in \Omega. \tag{2.4}$$

(Since Ω enjoys the exterior sphere condition, we can apply the result in [V].) Moreover, if u satisfies (1.4), then for any fixed $s > 0$, W is constant on ∂D. Indeed,

$$W(x,s) = s \int_0^{+\infty} a(t)\ e^{-s\ t} dt := A(s), \ \ x \in \partial D. \tag{2.5}$$

Thus, in view of the result in [V], we can define the positive number $R > 0$ by

$$R = \lim_{s \to +\infty} \{-\frac{1}{\sqrt{s}} \log A(s)\}. \tag{2.6}$$

In Lemma 2.1 below, we prove analyticity of ∂D and $\partial\Omega$ by using our balance law.

Lemma 2.1 *The following assertions hold:*

(i) *for every* $x \in \partial D$, $d(x) = R$, *where* d *is defined by (2.4);*

(ii) ∂D *is analytic;*

(iii) $\partial\Omega$ *is analytic and* $\partial\Omega = \{x \in \mathbb{R}^N : \text{dist}\,(x, D) = R\}$;

(iv) *the mapping:* $\partial D \ni x \mapsto y(x) \equiv x - R\nu^*(x) \in \partial\Omega$ *is a diffeomorphism, where* $\nu^*(x)$ *denotes the interior unit normal vector to* ∂D *at* $x \in \partial D$;

(v) *for every* $x \in \partial D$, $\nabla d(y(x)) = \nu^*(x)$ *and* $\overline{B_R(x)} \cap \partial\Omega = \{y(x)\}$;

(vi) *let* $\kappa_j(y)$, $j = 1, \dots, N-1$ *denote the* j*-th principal curvature at* $y \in \partial\Omega$ *of the analytic surface* $\partial\Omega$ *with respect to the interior normal direction to* $\partial\Omega$. *Then* $\kappa_j(y) < \frac{1}{R}$, $j = 1, \dots, N-1$, *for every* $y \in \partial\Omega$.

Proof. (i) The result in [V] and the definition (2.6) of R yield this assertion.

(ii) It suffices to show that, for every point $x \in \partial D$, there exists a time $t^* > 0$ such that $\nabla u(x, t^*) \neq 0$, since u is analytic with respect to the space variable.

Assume by contradiction that there exists a point $x_0 \in \partial D$ such that $\nabla u(x_0, t) = 0$ for every $t > 0$. Since u is continuous up to $\partial\Omega \times (0, +\infty)$, by Theorem 1.2 (ii), we can infer that

$$\int_{\partial B_R(x_0)} (x - x_0) \cdot u(x, t)\, dS_x = 0 \quad \text{for every } t > 0,$$

and hence

$$\int_{\partial B_R(x_0)} (x - x_0) \cdot W(x, s)\, dS_x = 0 \quad \text{for every } s > 0, \tag{2.7}$$

in view of (2.1).

On the other hand, since D satisfies the interior cone condition, there exists a finite right spherical cone K with vertex at x_0 such that $K \subset \overline{D}$ and $\overline{K} \cap \partial D = \{x_0\}$. By translating and rotating if needed, we can suppose that $x_0 = 0$ and that K is the set $\{x \in B_\rho(0) : x_N < -|x| \cos\theta\}$, where $\rho \in (0, R)$ and $\theta \in (0, \frac{\pi}{2})$.

Since $K \subset \overline{D}$ and $\overline{K} \cap \partial D = \{0\}$, proposition (i) implies that

$$d(x) > R \quad \text{for every } x \in K. \tag{2.8}$$

The set defined by

$$V = \{x \in \partial B_R(0) : x_N \geq R \sin\theta\}, \tag{2.9}$$

is such that

$$\partial\Omega \cap \partial B_R(0) \subset V, \tag{2.10}$$

because, otherwise, there would be a point in K contradicting (2.8).

Thus, from (2.10) it follows that we can choose a number $\delta > 0$ such that

$$d(x) \geq 5\delta \quad \text{for every } x \in \partial B_R(0) \cap \{x_N \leq 0\}. \tag{2.11}$$

Since we know that $-\frac{1}{\sqrt{s}} \log W(x,s)$ converges uniformly on $\overline{\Omega}$ to $d(x)$ as $s \to +\infty$, we can choose $s^* > 0$ such that

$$\left| -\frac{1}{\sqrt{s}} \log W(x,s) - d(x) \right| < \delta,$$

for every $x \in \overline{\Omega}$ and every $s \geq s^*$. This latter inequality, together with (2.9), (2.10), and (2.11), gives, for every $s \geq s^*$, the following two estimates:

$$\begin{aligned} &\int_{\partial B_R(0)\cap\{x_N\leq 0\}} x_N\, W(x,s)\, dS_x \geq -\tfrac{1}{2} R\, e^{-4\delta\sqrt{s}}\, \mathcal{H}^{N-1}\left(\partial B_R(0)\right), \\ &\int_{V\cap\overline{\Omega}_{2\delta}} x_N\, W(x,s)\, dS_x \geq R \sin\theta\, e^{-3\delta\sqrt{s}}\, \mathcal{H}^{N-1}\left(V \cap \overline{\Omega}_{2\delta}\right). \end{aligned} \tag{2.12}$$

Here $\mathcal{H}^{N-1}(\cdot)$ denotes the $(N-1)$–dimensional Hausdorff measure and $\Omega_{2\delta}$ is defined by

$$\Omega_{2\delta} = \{x \in \Omega : d(x) < 2\delta\}. \tag{2.13}$$

A consequence of (2.12) is that, for every $s \geq s^*$,

$$\begin{aligned} &\int_{\partial B_R(0)} x_N\, W(x,s)\, dS_x \geq \\ &\int_{V\cap\overline{\Omega}_{2\delta}} x_N\, W(x,s)\, dS_x + \int_{\partial B_R(0)\cap\{x_N\leq 0\}} x_N\, W(x,s)\, dS_x \geq \\ &R\, e^{-3\delta\sqrt{s}} \left[\sin\theta\, \mathcal{H}^{N-1}\left(V \cap \overline{\Omega}_{2\delta}\right) - \frac{1}{2} e^{-\delta\sqrt{s}}\, \mathcal{H}^{N-1}\left(\partial B_R(0)\right) \right]. \end{aligned}$$

Therefore, we obtain a contradiction by observing that the first term of this chain of inequalities equals zero, by (2.7), while the last term can be made positive by choosing $s > 0$ sufficiently large.

(iii), (iv), and (v) Let

$$\Gamma = \{x \in \mathbb{R}^N : \text{dist}\,(x, D) = R\}.$$

It is clear that $\Gamma \subset \partial\Omega$. Take any point $x \in \partial D$. Then, there exists a unique point $y \in \partial\Omega$ such that $\overline{B_R(x)} \cap \partial\Omega = \{y\}$. Indeed, since ∂D is analytic by (ii), if $\tilde{y} \in \overline{B_R(x)} \cap \partial\Omega$ and $\tilde{y} \neq y$, then

$$\frac{y-x}{|y-x|} = -\nu^*(x) = \frac{\tilde{y}-x}{|\tilde{y}-x|},$$

where $\nu^*(x)$ is the interior unit normal vector to ∂D at x — a contradiction. Since Ω enjoys the exterior sphere property, there exists a ball $B_r(z)$ such that $\overline{B_r(z)} \cap \overline{\Omega} = \{y\}$, and hence $\overline{B_r(z)} \cap \overline{B_R(x)} = \{y\}$. Therefore,

$$\text{dist }(z, D) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \overline{D} = \{x\}. \tag{2.14}$$

Let κ_j^*, $j = 1, \ldots, N-1$, denote the principal curvatures of the surface ∂D with respect to the interior normal direction to ∂D. Then (2.14) implies that

$$\kappa_j^*(x) \geq -\frac{1}{r+R}, \ j = 1, \ldots, N-1.$$

Since $\kappa_j^* > -\frac{1}{R}$ on ∂D, for every $j = 1, \ldots, N-1$, Γ is an analytic hypersurface diffeomorphic to ∂D (see [GT], Lemma 14.16), and hence Γ equals $\partial\Omega$. Assertions (iii), (iv), and (v) then follow at once.

(vi) Take any point $y \in \partial\Omega$. Propositions (iii) and (iv) imply that there exists a unique $x \in \partial D$ such that $\overline{B_R(y)} \cap \overline{D} = \{x\}$. Since ∂D is analytic, D satisfies the interior sphere condition, that is there exists a ball $B_r(z) \subset D$ such that $\overline{B_r(z)} \cap \partial D = \{x\}$. Therefore,

$$d(z) = r + R \quad \text{and} \quad \overline{B_{r+R}(z)} \cap \partial\Omega = \{y\}, \tag{2.15}$$

and consequently

$$\kappa_j(y) \leq \frac{1}{r+R}, \ j = 1, \ldots, N-1.$$

Assertion (vi) is proved. □

Let us show that the two functions

$$W_\varepsilon^{\pm}(x,s) = \exp\{-\sqrt{s(1 \mp \varepsilon)}\ d(x)\}, \tag{2.16}$$

where $d(x)$ is defined by (2.4), provide respectively an upper and a lower barrier for W in Ω for large values of s.

Lemma 2.2 *For every $\varepsilon > 0$, there exists a positive number s_ε such that*

$$W_\varepsilon^-(x,s) \leq W(x,s) \leq W_\varepsilon^+(x,s) \tag{2.17}$$

for every $x \in \overline{\Omega}$ and every $s \geq s_\varepsilon$.

Proof. Choose a number $\delta > 0$ such that the function $d = d(x)$ defined in (2.4) is of class C^2 in the set $\overline{\Omega_\delta}$ where

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}. \tag{2.18}$$

Let $W_\varepsilon^{\pm}(x,s)$ be given by (2.16). A straightforward computation gives

$$\Delta W_\varepsilon^{\pm} - s\, W_\varepsilon^{\pm} = \mp\varepsilon\sqrt{s}\ \{\sqrt{s} \pm \frac{\sqrt{(1\mp\varepsilon)}}{\varepsilon}\ \Delta d\} W_\varepsilon^{\pm} \quad \text{in } \ \Omega_\delta.$$

Set $M_\delta = \max\limits_{\overline{\Omega}_\delta} |\Delta d|$. If $s \geq \frac{1+\varepsilon}{\varepsilon^2}\ M_\delta^2$, then

$$\left.\begin{array}{l} \Delta W_\varepsilon^{+} - s\, W_\varepsilon^{+} \leq 0 \\ \Delta W_\varepsilon^{-} - s\, W_\varepsilon^{-} \geq 0 \end{array}\right. \quad \text{in } \ \Omega_\delta. \tag{2.19}$$

Since the function $-\frac{1}{\sqrt{s}}\log W(x,s)$ converges uniformly on $\overline{\Omega}$ to $d(x)$ as $s \to +\infty$, there exists a number $s^* > 0$ such that

$$-\delta(1-\sqrt{1-\varepsilon}) \leq -\frac{1}{\sqrt{s}}\log W(x,s) - d(x) \leq \delta(\sqrt{1+\varepsilon}-1), \ \ x \in \overline{\Omega},$$

for every $s \geq s^*$. Hence, since $d(x) \geq \delta$ for every $x \in \Omega \setminus \Omega_\delta$, we obtain

$$W_\varepsilon^{-}(x,s) \leq W(x,s) \leq W_\varepsilon^{+}(x,s), \quad x \in \Omega \setminus \Omega_\delta, \tag{2.20}$$

for every $s \geq s^*$. Moreover,

$$W_\varepsilon^{-}(x,s) = W(x,s) = W_\varepsilon^{+}(x,s) = 1, \quad x \in \partial\Omega, \tag{2.21}$$

for every $s > 0$, clearly.

Choose $s_\varepsilon = \max(s^*, \frac{1+\varepsilon}{\varepsilon^2}\ M_\delta^2)$. Then by the comparison principle, from (2.19), (2.20) and (2.21), we have

$$W_\varepsilon^{-}(x,s) \leq W(x,s) \leq W_\varepsilon^{+}(x,s), \quad x \in \Omega_\delta, \tag{2.22}$$

for every $s \geq s_\varepsilon$. Combining (2.22) with (2.20) yields (2.17). □

With the help of Lemma 2.1, we obtain

Lemma 2.3 *Let $x_0 \in \partial D$ and put $y_0 = y(x_0) \in \partial\Omega$, where $y(x_0)$ is given in Lemma 2.1 (see (iv) and (v)). Then*

$$\lim_{s\to+\infty} s^{\frac{N-1}{4}} \int\limits_{\partial B_R(x_0)} e^{-\sqrt{s(1\pm\varepsilon)}\, d(x)}\, dS_x = \left(\frac{2\pi}{\sqrt{1\pm\varepsilon}}\right)^{\frac{N-1}{2}} \left\{\prod_{j=1}^{N-1}[\frac{1}{R} - \kappa_j(y_0)]\right\}^{-\frac{1}{2}}, \tag{2.23}$$

where $\kappa_j(y)$, $j = 1, \dots, N-1$ denotes the j-th principal curvature at $y \in \partial\Omega$ of the analytic surface $\partial\Omega$ with respect to the interior normal direction to $\partial\Omega$.

Proof. In view of proposition (vi) of Lemma 2.1, in order to prove this lemma we can use Laplace's method (see [deB], p. 71 for example) or the stationary phase method (see [Ev], pp. 208 – 217 for example). See [MS 3] for details. □

Combining Lemma 2.3 with Lemma 2.2 yields

Lemma 2.4 *Let $x_0 \in \partial D$ and put $y_0 = y(x_0) \in \partial\Omega$, where $y(x_0)$ is given in Lemma 2.1 (see (iv) and (v)). Then*

$$\lim_{s\to+\infty} s^{\frac{N-1}{4}} \int\limits_{\partial B_R(x_0)} W(x,s)\, dS_x = (2\pi)^{\frac{N-1}{2}} \left\{\prod_{j=1}^{N-1}[\frac{1}{R} - \kappa_j(y_0)]\right\}^{-\frac{1}{2}}. \tag{2.24}$$

The last lemma is

Lemma 2.5 *We have*

$$\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j(y)\right] = \text{a constant} > 0, \quad \textit{for every } y \in \partial\Omega, \tag{2.25}$$

where $\kappa_j(y)$, $j = 1, \dots, N-1$ denotes the j-th principal curvature at $y \in \partial\Omega$ of the analytic surface $\partial\Omega$ with respect to the interior normal direction to $\partial\Omega$. In particular, if $N = 2$, Ω must be a ball.

Proof. Let p and q be two distinct points in $\partial\Omega$. Propositions (iv) and (v) from Lemma 2.1 guarantee that there exist two distinct points P, Q in ∂D such that $p = y(P)$ and $q = y(Q)$ in (iv).

For $x \in B_R(0)$, consider the function

$$v(x,t) = u(x+P,t) - u(x+Q,t). \tag{2.26}$$

Then $v = v(x,t)$ satisfies the heat equation in $B_R(0) \times (0,+\infty)$ and by (1.4)

$$v(0,t) = u(P,t) - u(Q,t) = 0,$$

for every $t > 0$. Since v is continuous up to $\partial B_R(0) \times (0,+\infty)$, by Theorem 1.2 (i) we obtain

$$\int\limits_{\partial B_R(0)} v(x,t)\ dS_x = 0$$

for every $t > 0$, and hence

$$\int\limits_{\partial B_R(P)} u(x,t)\ dS_x = \int\limits_{\partial B_R(Q)} u(x,t)\ dS_x$$

for every $t > 0$. Therefore, in view of (2.1), we have

$$\int\limits_{\partial B_R(P)} W(x,s)\ dS_x = \int\limits_{\partial B_R(Q)} W(x,s)\ dS_x \tag{2.27}$$

for every $s > 0$. With the help of Lemma 2.4, by multiplying both sides of (2.27) by $s^{\frac{N-1}{4}}$, we can take the limits as $s \to +\infty$. Therefore, since $p = y(P)$ and $q = y(Q)$, after some manipulation, we obtain:

$$\prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(p)\right] = \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(q)\right],$$

that is, (2.25) holds. □

We quote A.D. Aleksandrov's uniqueness theorem from [Alek], p. 412, adjusted to our notations. A special case of this theorem is the well-known *Soap-Bubble Theorem* (see also [R]).

Theorem 2.6 (Aleksandrov) *Let $\Phi = \Phi(\kappa_1, \cdots, \kappa_{N-1})$ be a continuously differentiable function, defined for $\kappa_1 \geq \cdots \geq \kappa_{N-1}$, and subject to the condition $\frac{\partial \Phi}{\partial \kappa_i} > 0$ ($i = 1, \cdots, N-1$).*

Suppose that in $\mathbb{R}^N$ we have a twice-differentiable closed surface S without self-intersections and with bounded principal curvatures.

If on the surface S the function Φ of its principal curvatures $\kappa_1, \cdots, \kappa_{N-1}$ has at all points one and the same value, then S is a sphere.

Proof of Theorem 1.1. By Lemma 2.5, it suffices to consider the case where $N \geq 3$. We set

$$\Phi = \Phi(\kappa_1, \cdots, \kappa_{N-1}) = -\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j\right] \tag{2.28}$$

and observe that

$$\frac{\partial \Phi}{\partial \kappa_i} > 0 \quad (i = 1, \cdots, N-1), \text{ if } \max_{1 \leq j \leq N-1} \kappa_j < \frac{1}{R}.$$

Since condition (2.25) holds by Lemma 2.5, we infer that the function Φ is constant on $\partial\Omega$. Therefore, by applying Theorem 2.6 to each connected component of $\partial\Omega$, we conclude that $\partial\Omega$ must be a sphere. □

Remark. The method of proof of Theorem 2.6 is called *Aleksandrov's reflection principle* or *the method of moving planes*, which is based on the maximum principle for elliptic partial differential equations of second order. In fact, by using local coordinates, the condition $\Phi(\kappa_1, \ldots, \kappa_{N-1}) =$ constant on the surface S can be converted into a second order partial differential equation which is of elliptic type, since $\frac{\partial \Phi}{\partial \kappa_i} > 0$ $(i = 1, \cdots, N-1)$. In the case the function Φ is given by (2.28), we obtain an equation of Monge-Ampère type.

3 Concluding remarks

By the same method as in the proof of Theorem 1.1, we see that the following theorem also holds.

Theorem 3.1 *Let Ω be an exterior domain in $\mathbb{R}^N$, $N \geq 2$, satisfying the exterior sphere condition and suppose that D is an exterior domain, with boundary ∂D, satisfying the interior cone condition, and such that $\overline{D} \subset \Omega$.*

Assume that the solution u to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0, +\infty) \to (0, +\infty)$.

Then $\partial\Omega$ must be a sphere. That is, Ω must be the exterior of a ball.

Since both $\partial\Omega$ and ∂D are compact, it follows from the barrier arguments with the help of Varadhan's result that inequality (2.17) holds for x in an arbitrary bounded neighborhood of $\partial\Omega$ and for sufficiently large s. Therefore, we get the same relation of the principal curvatures of $\partial\Omega$. Hence each connected component of $\partial\Omega$ is a sphere with the same radius. Moreover, by analyticity, $u(x,t)$ must be radially symmetric in x with respect to each center of each connected component of $\partial\Omega$. Thus $\partial\Omega$ must be a sphere.

Professor Messoud A. Efendiev gave us the following conjecture:

Consider domains Ω whose boundary $\partial\Omega$ is not compact. In particular, let Ω be a unbounded domain above a Lipschitz graph $x_N = \varphi(x_1, \ldots, x_{N-1})$ over $\mathbb{R}^{N-1}$. Suppose that there exists an invariant isothermic surface. Then $\partial\Omega$ must be a hyperplane.

Our answer to this conjecture is the following theorem:

Theorem 3.2 *Let Ω be a unbounded domain above a locally Lipschitz graph $x_N = \varphi(x_1, \ldots, x_{N-1})$ over $\mathbb{R}^{N-1}$ such that*

$$\nabla\varphi(x) = o(|x|^{\frac{1}{2}}) \text{ near infinity.} \tag{3.1}$$

Suppose that Ω satisfies the uniform exterior sphere condition, that is, there exists $r > 0$ such that for every $x \in \partial\Omega$ there exists a ball $B_r(z)$ with $\overline{B_r(z)} \cap \overline{\Omega} = \{x\}$. Assume that there exists a domain D with $\overline{D} \subset \Omega$ such that the solution u to problem (1.1)-(1.3) satisfies the condition (1.4) for some function $a : (0, +\infty) \to (0, +\infty)$.

Then $\partial\Omega$ must be a hyperplane.

With the help of curvature estimates in a Bernstein's theorem due to L. Caffarelli, L. Nirenberg, and J. Spruck (see Theorem 2" and its proof in [CNS]), we can prove this theorem. The details will be given in a forthcoming paper.

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